A New Approach to Euler Splines

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Starting from the exponential Euler polynomials discussed by Euler in "Institutionis Calculi Differentialis," Vol. II, 1755, the author introduced in "Linear operators and approximation," Vol. 20, 1972, the so-called *exponential Euler splines*. Here we describe a new approach to these splines. Let t be a constant such that

$$t = t\, e^{i u}, \qquad -\pi < u < \pi, t \neq 0, t \neq 1.$$

Let $S_1(x; t)$ be the cardinal linear spline such that

$$S_1(v; t) = t^v \qquad \text{for all} \quad v \in Z.$$

Starting from $S_1(x; t)$ it is shown that we obtain all higher degree exponential Euler splines recursively by the averaging operation

$$S_n(x; t) = \int_{v-1/2}^{v+1/2} S_{n-1}(u; t)\, du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u; t)\, du \qquad (n = 2, 3,\ldots).$$

Here $S_n(x; t)$ is a cardinal spline of degree n if n is odd, while $S_n(x + \tfrac{1}{2}; t)$ is a cardinal spline if n is even. It is shown that they have the properties

$$S_n(v; t) = t^v \qquad \text{for} \quad v \in Z,$$

$$\lim_{n \to \infty} S_n(x; t) = t^x = |t|^x e^{i u x}.$$

1. INTRODUCTION

Here I wish to describe a new recursive construction of the Exponential Euler splines which I introduced in [2]—a construction to be described in Section 2. We first recall their definition as given in [2].

Let $\mathscr{S}_n = \{S(x)\}$ denote the class of *cardinal splines* $S(x)$ of degree n, having knots at the integers v. Furthermore, let $\mathscr{S}_n^* = \{S(x + \tfrac{1}{2}) \in \mathscr{S}_n\}$ be

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the class of *midpoint cardinal splines*, i.e., having knots at $v - \frac{1}{2}$. For a constant t such that

$$t = |t| e^{i\alpha}, \qquad -\pi < \alpha < \pi, \, t \neq 0, \, t \neq 1, \tag{1.1}$$

we wish to construct the cardinal spline which interpolates at all integers the exponential function $t^x = |t|^x e^{i\alpha x}$. In $|2|$ this problem was solved within each of the two classes $\mathscr{S}_n$ and $\mathscr{S}_n^*$. These solutions $s_n(x)$ and $s_n^*(x)$ were constructed as follows:

DEFINITION. We define $s_n(x) \in \mathscr{S}_n$ $(n \geqslant 1)$ in two steps. We start from Euler's generating function $|1,$ Chap. VII, Sect. 178$|$

$$\frac{t-1}{t-e^z} e^{xz} = \sum_0^{\infty} \frac{A_n(x; t)}{n!} z^n, \tag{1.2}$$

defining the Eulerian polynomial $A_n(x; t)$, set[1]

$$p_n(x) = A_n(x; t)/A_n(0; t), \tag{1.3}$$

and define

$$s_n(x) = p_n(x) \qquad \text{if} \quad 0 \leqslant x < 1. \tag{1.4}$$

The second step consists of extending the definition of $s_n(x)$ to all real x by means of the functional equation

$$s_n(x + 1) = t s_n(x). \tag{1.5}$$

The remarkable property of the polynomial $p_n(x)$, defined by (1.3), *is that the resulting $s_n(x) \in C^{n-1}(R)$ and therefore $s_n(x) \in \mathscr{S}_n$.* From (1.3), $p_n(0) = 1$ and (1.5) shows that $s_n(v) = t^v$ for all $v \in Z$. The function $s_n(x)$ is the *exponential Euler spline of the class $\mathscr{S}_n$.* Euler provided in 1755 the seed (1.3); all that I had to do in $|2|$ to obtain $s_n(x)$ was to extend this seed $p_n(x)$ to all x by means of (1.5).

DEFINITION. We define $s_n^*(x) \in \mathscr{S}_n^*$. Now we use the generating function

$$\frac{t-1}{t-e^z} e^{(x+1/2)z} = \sum_0^{\infty} \frac{B_n(x; t)}{n!} z^n, \tag{1.6}$$

[1] It is known that if $A_n(0, t) = 0$, then t is negative, which is excluded in (1.1); Similarly for $B_n(0, t)$ in (1.7).

defining the midpoint Eulerian polynomials $B_n(x; t)$, set

$$p_n^*(x) = B_n(x; t)/B_n(0; t),$$ (1.7)

and define

$$s_n^*(x) = p_n^*(x) \qquad \text{if} \quad -\tfrac{1}{2} \leqslant x < \tfrac{1}{2}.$$ (1.8)

The second step is to extend $s_n^*(x)$ to all real x by

$$s_n^*(x + 1) = t s_n^*(x).$$ (1.9)

Now (1.7) is such that $s_n^*(x) \in C^{n-1}(R)$ and therefore $s_n^*(x) \in \mathscr{S}_n^*$.

   In Section 2 we define recursively, by (2.1) and (2.5), a sequence of cardinal splines

$$S_n(x) = S_n(x; t) \qquad (n = 1, 2, ...)$$ (1.10)

such that

$$\begin{aligned} S_n(x) &= s_n(x) \qquad \textit{if} \quad n \text{ is odd,} \\ &= s_n^*(x) \qquad \textit{if} \quad n \text{ is even.} \end{aligned}$$ (1.11)

In the remainder of the paper we use only the new construction to obtain all known properties of the Euler splines and also some new ones.

   While our assumptions (1.1) excluded negative values of t, we use in Section 6 the new construction to discuss the classical case when $t = -1$.

2. A NEW APPROACH TO THE EXPONENTIAL EULER SPLINES

   As t is kept fixed throughout, we simplify our notation by writing $S_n(x; t) = S_n(x)$. *We define $S_1(x)$ by the conditions*

$$S_1(x) \in \mathscr{S}_1, \qquad S_1(v) = t^v \qquad (v \in Z).$$ (2.1)

Plotting the sequence (t^v) in the complex plane we see that the graph of $z = S_1(x)$ $(-\infty < x < \infty)$ *is the biinfinite polygon with successive vertices* (t^v). Equivalently, we may define $S_1(x)$ by setting

$$S_1(x) = 1 + (t - 1)x \qquad \text{if} \quad 0 \leqslant x \leqslant 1,$$ (2.2)

and extend its definition to all real x by the functional equation

$$S_1(x + 1) = t S_1(x) \qquad (x \in R).$$ (2.3)

Starting from $S_1(x)$, we define the exponential Euler spline

$$S_n(x) = S_n(x; t) \qquad (x \in R, n = 2, 3,...)$$ (2.4)

recursively by the averaging operation

$$S_n(x) = \int_{x-1/2}^{x+1/2} S_{n-1}(u)\, du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u)\, du \quad (n = 2, 3,....).$$ (2.5)

It does seem remarkable that starting from the rough polygon of $S_1(x)$, the repeated smoothing operation

$$\int_{x-1/2}^{x+1/2} (\cdot)\, du \Big/ \int_{-1/2}^{1/2} (\cdot)\, du$$

should produce the progressively smoother sequence of interpolants $S_n(x)$. Also that our formula (2.5) does not depend on t explicitly.

In Section 3 we state the properties of $S_n(x)$, which will also show the validity of their definition (2.5). These properties are established in Sections 4 and 5.

3. VALIDITY OF THE DEFINITION OF THE $S_n(x)$ AND THEIR PROPERTIES

Their definition is evidently valid in the case that $t > 0$, for $S_1(x) > 0$ $(x \in R)$, and by induction we see that

$$S_n(x) > 0 \qquad \text{for all} \quad x \text{ and } n \ (t > 0).$$

We loose no generality by assuming in Sections 3–5 that in (1.4) we have

$$0 < \alpha < \pi.$$ (3.1)

In this case the following propositions will be established.

(I) *We have*

$$I_n(x) := \int_{x-1/2}^{x+1/2} S_n(u)\, du \neq 0 \qquad \text{for} \quad x \in R, n = 1, 2.....$$ (3.2)

This justifies our definition (2.5) since no denominators can vanish.

(II) $S_n(x)$ *satisfies the functional equation*

$$S_n(x + 1) = t S_n(x) \qquad \text{for} \quad x \in R, n = 1, 2,....$$ (3.3)

(III)   *Along the arc*

$$A_{n,x}: z = S_n(u) \qquad (x - \tfrac{1}{2} \leqslant u \leqslant x + \tfrac{1}{2}) \tag{3.4}$$

the argument of $z = S_n(u)$ *is steadily increasing by the amount*

$$\arg S_n(x + \tfrac{1}{2}) - \arg S_n(x - \tfrac{1}{2}) = \alpha. \tag{3.5}$$

(see Fig. 1).

   The *tangent vector* $S_n'(u)$ *is steadily turning counter-clockwise along* $A_{n,x}$ *from* $S_n'(x - \tfrac{1}{2})$ *to* $S_n'(x + \tfrac{1}{2})$, *such that*

$$\arg S_n'(x + \tfrac{1}{2}) - \arg S_n'(x - \tfrac{1}{2}) = \alpha. \tag{3.6}$$

Denoting by T *the intersection of the lines carrying the vectors* $S_n'(x \pm \tfrac{1}{2})$, *the four points*

$$0, \; S_n(x + \tfrac{1}{2}), \; T, \; S_n(x - \tfrac{1}{2}) \qquad \textit{are on a circle } \Gamma. \tag{3.7}$$

The arc $A_{n,x}$ *is contained in the quadrilateral* Q *of the four points* (3.7).

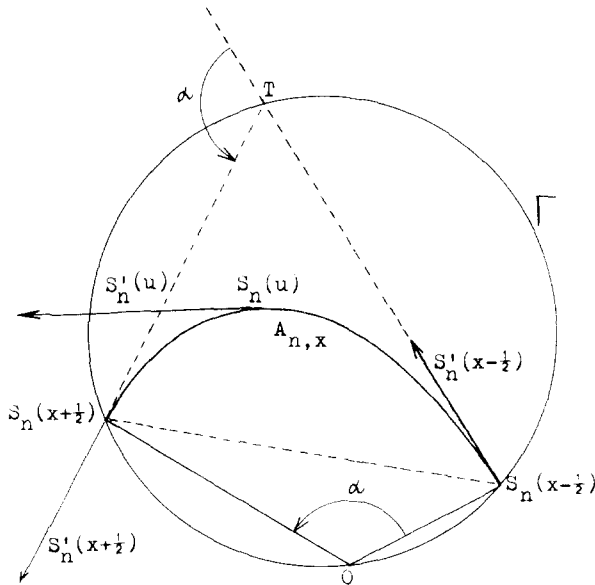$$\textit{The arc } A_{n,x} \textit{ is convex (Fig. 1).} \tag{3.8}$$

FIGURE 1

(IV)   *For all* n,

$$S_n(x) \in \begin{cases} \mathscr{S}_n & \text{if } n \text{ is odd,} \\ \mathscr{S}_n^* & \text{if } n \text{ is even.} \end{cases} \qquad (3.9)$$

V.   *The limit relation*

$$\lim_{n \to \infty} S_n(x) = t^x = |t|^x e^{i a x} \qquad (3.10)$$

holds uniformly in x in every finite interval of R.

   (VI)   *For the special case so far excluded,*

$$t = -1, \qquad (3.11)$$

propositions (II) *and* (IV) *remain valid, while* (I) *is to be replaced by*

$$\int_{-1/2}^{1/2} S_n(u; -1) \, du > 0 \qquad (n = 1, 2,....), \qquad (3.12)$$

and (v) *by*

$$\lim_{n \to \infty} S_n(x; -1) = \cos \pi x. \qquad (3.13)$$

4. PROOFS OF PROPOSITIONS (I)–(IV)

   All these propositions are evident if $n = 1$ in view of (2.3), which implies that

   triangle $(0, S_1(x - \tfrac{1}{2}), S_1(x + \tfrac{1}{2}))$ is similar to triangle $(0, 1, t)$.

Notice also that if $n = 1$, the arc $A_{1,x}$ of (3.4) is identical with the union of the two sides of the quadrilateral Q of Fig. 1 that meet in T.

   Assume that (I), (II), (III), are already established for all values up to and including $n - 1$ and let us prove them for the value n.

   (II) Using the notation of (3.2), we have

$$S_n(x + 1) = \int_{x + 1/2}^{x + 3/2} S_{n-1}(u)/I_{n-1}(0)$$

$$= \int_{x - 1/2}^{x + 1/2} S_{n-1}(u + 1) \, du/I_{n-1}(0) \qquad (4.1)$$

$$= \int_{x - 1/2}^{x + 1/2} t S_{n-1}(u) \, du/I_{n-1}(0) = t S_n(x).$$

(III) Relation (3.5) follows immediately from (3.3) by replacing x by $x - \frac{1}{2}$. The remaining statements are obtained as follows: In (2.5), we replace x by u and differentiate with respect to u obtaining

$$S'_n(u) = \{S_{n-1}(u + \tfrac{1}{2}) - S_{n-1}(u - \tfrac{1}{2})\}/I_{n-1}(0)$$
$$= \{tS_{n-1}(u - \tfrac{1}{2}) - S_{n-1}(u - \tfrac{1}{2})\}/I_{n-1}(0),$$

and finally

$$S'_n(u) = c \cdot S_{n-1}(u - \tfrac{1}{2}), \text{ where } c = (t - 1)/I_{n-1}(0) \,(\neq 0). \qquad (4.2)$$

Since $S_{n-1}(u - \frac{1}{2})$ turns counter-clockwise (Fig. 1) by the angle α as u increases from $x - \frac{1}{2}$ to $x + \frac{1}{2}$, we see that (4.2) proves that $S'_n(u)$ turns, by (3.6), by the angle α, hence (3.6) holds. This already proves all our statements for n, as summarized in our Fig. 1.

(I) Let x' be such that $x - \frac{1}{2} < x' < x + \frac{1}{2}$ and

$$\arg S(x') = \tfrac{1}{2}\{\arg S_n(x + \tfrac{1}{2}) + \arg S_n(x - \tfrac{1}{2})\}.$$

It follows from (3.5) that

$$-\frac{\alpha}{2} \leqslant \arg S_n(u) - \arg S_n(x') \leqslant \frac{\alpha}{2} \quad \left(x - \frac{1}{2} \leqslant u \leqslant x + \frac{1}{2}\right). \quad (4.3)$$

Dividing (3.2) by $S_n(x') = |S_n(x')| \exp(i \arg S_n(x'))$; we obtain

$$I_n(x)/S_n(x') = \int_{x-1/2}^{x+1/2} |S_n(u)/S_n(x')| \, e^{i(\arg S_n(u) - \arg S_n(x'))} \, du,$$

and from (4.3) we conclude that

$$\operatorname{Re}\{I_n(x)/S_n(x')\} \geqslant \int_{x-1/2}^{x+1/2} |S_n(u)/S_n(x')| \cdot \cos\frac{\alpha}{2} \, du > 0.$$

This proves (3.2).

(IV) This is evident by induction from (2.5) because the limits of integration are $x \pm \frac{1}{2}$, in view of the general properties of cardinal splines. It liewise follows by induction that $S_n(x) \in C^{n-1}(R)$ for all n.

5. Proof of Proposition (V)

We write

$$\gamma = \log t = \log |t| + i\alpha, \qquad (5.1)$$

hence $t = e^\gamma$, and define the new function $\omega_n(x)$ by setting

$$S_n(x) = t^x \omega_n(x) \qquad (x \in R). \tag{5.2}$$

From (3.3) we have that

$$S_n(x + 1) = t^{x+1} \omega_n(x + 1) = t S_n(x) = t t^x \omega_n(x),$$

hence $\omega_n(x + 1) = \omega_n(x)$, showing that $\omega_n(x)$ is periodic of period 1. Using (2.2) let us find the Fourier series of

$$\omega_1(x) = (1 + (t - 1)x) e^{-\gamma x} \qquad \text{if} \quad 0 \leqslant x \leqslant 1. \tag{5.3}$$

Since $\omega_1(x) \in C(R)$ its Fourier series should converge absolutely.

For its Fourier coefficients we find by integration by parts that

$$a_v = \frac{1}{2\pi} \int_0^1 \omega_1(x) e^{-2\pi i v x} \, dx = \frac{(t-1)^2}{2\pi t} \frac{1}{(\gamma + 2\pi i v)^2}$$

and so

$$S_1(x) = e^{\gamma x} \omega_1(x) = \frac{(t-1)^2}{2\pi t} \sum_{-\infty}^{\infty} \frac{e^{(\gamma + 2\pi i v)x}}{(\gamma + 2\pi i v)^2}. \tag{5.4}$$

Next we observe that

$$\int_{x-1/2}^{x+1/2} e^{(\gamma + 2\pi i v)u} \, du = (-1)^v \frac{e^{\gamma/2} - e^{-\gamma/2}}{\gamma + 2\pi i v} e^{(\gamma + 2\pi i v)x}. \tag{5.5}$$

Performing the operation $\int_{x-1/2}^{x+1/2}$ $n - 1$ times on (5.4), and dividing the result by its value at $x = 0$ to enforce $S_n(0) = 1$, we obtain the expansion

$$S_n(x) = \sum_{-\infty}^{\infty} \frac{(-1)^{v(n-1)}}{(\gamma + 2\pi i v)^{n+1}} e^{(\gamma + 2\pi i v)x} \Bigg/ \sum_{-\infty}^{\infty} \frac{(-1)^{v(n+1)}}{(\gamma + 2\pi i v)^{n+1}}. \tag{5.6}$$

Since $t^x = e^{\gamma x}$, we find for $\omega_n(x)$ of (5.2) the expansion

$$\omega_n(x) = \sum_{-\infty}^{\infty} \frac{(-1)^{v(n-1)}}{(\gamma + 2\pi i v)^{n+1}} e^{2\pi i v x} \Bigg/ \sum_{-\infty}^{\infty} \frac{(-1)^{v(n+1)}}{(\gamma + 2\pi i v)^{n+1}}. \tag{5.7}$$

Let us prove that

$$\lim_{n \to \infty} \omega_n(x) = 1. \tag{5.8}$$

Proof. We multiply both terms of the fraction (5.7) by γ^{n+1}; except for the two terms for $v = 0$, which are $=1$ in both series, all other terms are in

absolute value $= |\gamma/(\gamma + 2\pi i v)|^{n+1}$ $(v \neq 0)$, except for a constant factor. Writing $\rho := \log |t|$, we have $\gamma = \rho + i\alpha$ and so

$$|\gamma/(\gamma + 2\pi i v)|^2 = |(\rho + i\alpha)/(\rho + i(\alpha + 2\pi v))|^2$$

$$= \frac{\rho^2 + \alpha^2}{\rho^2 + (\alpha + 2\pi v)^2}.$$

Now our condition $-\pi < \alpha < \pi$ implies that

$$|\alpha + 2\pi v| \geqslant 2\pi - |\alpha| > |\alpha| \qquad \text{if} \quad v \neq 0, \tag{5.9}$$

and so

$$\max_v |\gamma/(\gamma + 2\pi i v)| = \left(\frac{\rho^2 + \alpha^2}{\rho^2 + (2\pi - |\alpha|)^2} \right)^{1/2} = \delta_t < 1. \tag{5.10}$$

Moreover $\delta_t > 0$, since ρ and α cannot both vanish because $t \neq 1$. But then it is clear that

$$\omega_n(x) = 1 + O(\delta_t^{n+1}). \tag{5.11}$$

From (5.2) and (5.11) we conclude that

$$\frac{S_n(x)}{t^x} - 1 = O(\delta_t^{n+1}), \qquad \text{uniformly for } x \in R, \tag{5.12}$$

where δ_t is defined by (5.10).

Notice that δ_t approaches 1 if α is close to $\pm\pi$.

On Programming the Exponential Function on a Computer

The construction of $s_n(x)$ as given by our Eqs. (1.2)–(1.5) seems eminently suitable to achieve the title of this subsection. Obtaining the polynomial $p_n(x)$ of (1.3) requires the following algorithm:

Determine the $a_r = a_r(t)$ recursively from the relations

$$1 + \binom{v}{1} a_1 + \binom{v}{2} a_2 + \cdots + a_v = t a_v \qquad (v = 1, 2, ...), \tag{5.13}$$

and then

$$p_n(x) = \left\{ x^n + \binom{n}{1} a_1 x^{n-1} + \binom{n}{2} a_2 x^{n-2} + \cdots + a_n \right\} / a_n. \tag{5.14}$$

This is particularly convenient if $t = 2$ when (5.1) reduces to

$$a_v = 1 + \binom{v}{1} a_1 + \cdots + \binom{v}{v-1} a_{v-1} \qquad (v = 1, 2, ..., a_0 = 1), \tag{5.15}$$

showing that all coefficients a_i are integers. Besides, the exponential 2^x is in several ways remarkable, since $\varDelta 2^x = 2^x$, and *among all entire functions that assume integer values for* $x = 0, 1, 2, ..., 2^x$ *is the one of least growth* (theorem of Hardy and Pólya).

If $t = 2$, then the ρ in (5.10) becomes $\log 2$; also $\alpha = 0$ and (5.10) shows that

$$\delta_2 = \left\{ 1 + \frac{4\pi^2}{(\log 2)^2} \right\}^{-1/2} = 0.1096.$$

This shows that $s_n(x) = s_n(x; 2)$ approximates 2^x to nearly n decimal places. In $|2, \text{p. } 403|$ we found that, for $t = 2$,

$$p_7(x) = (x^7 + 7x^6 + 63x^5 + 455x^4 + 2625x^3 + 11361x^2$$
$$+ 32781x + 47293)/47293.$$

Computations show that

$$|2^x - p_7(x)| < 10^{-7} \quad \text{in} \quad 0 \leqslant x \leqslant 1,$$

and that

the error function $2^x - p_7(x)$ *changes sign just once.* $\qquad$ (5.16)

An algorithm producing 2^x of course allows us to compute $e^x = 2^{x \log 2}$.

A better approximation of 2^x is obtained if we replace in $|0, 1|$ our polynomial $p_7(x)$ by the polynomial $*p_7(x)$ of best approximation of 2^x, afterwards obtaining the approximation $*s_7(x)$ for all real x by $*s_7(x + 1) = 2*s_7(x)$. However, *the new global approximation* $*s_7(x)$ *so obtained is evidently discontinuous for all integer values of* x, *while our approximation* $s_7(x)$ *is in the class* $C^6(-\infty, \infty)$.

Of course, the approximation (5.12) deteriorates as we pass from $t = 2$ to larger values of t. Thus we find that

$$\delta_e = (1 + 4\pi^2)^{-1/2} = 0.1572, \qquad \delta_{10} = \left(1 + \frac{4\pi^2}{(\log 10)^2} \right)^{-1/2} = 0.3441.$$

The remark (5.16) is not an isolated result, as we have proposition

(VII) (1) *If* $t > 0$, *then the two curves of the* (x, y) *plane*

$$\Gamma_n : y = S_n(x; t), \qquad \Gamma : y = t^x \qquad (x \in R), \qquad (5.17)$$

agree at the points $(x, y) = (v, t^v)$, *the slopes of the tangents to* Γ_n *being in a fixed nonvanishing ratio to the slopes of the corresponding tangents to* Γ. *It follows that* Γ_n *crosses* Γ *in every interval* $v < x < v + 1$.

(2)   *If in (1.4) we have $0 < \alpha < \pi$, then for the two curves in C*

$$\tilde{\Gamma}_n : z = S_n(x; t), \qquad \tilde{\Gamma} : z = t^x \qquad (x \in R),\qquad (5.18)$$

we have a similar situation: Either $\tilde{\Gamma}_n$ crosses $\tilde{\Gamma}$ in the same direction at all points t^v, or else $\tilde{\Gamma}_n$ is tangent to $\tilde{\Gamma}$ at all points t^v, as in Fig. 2, where $|t| = 1$.

Proof.   We drop the t, writing $S_n(x; t) = S_n(x)$, $t^x = S_\infty(x)$. From the relations $S'_n(x + 1) = tS'_n(x)$, $S'_\infty(x + 1) = tS'_\infty(x)$, we obtain, for $x = 0$, that

$$\frac{S'_n(v)}{S'_\infty(v)} = \frac{S'_n(0)}{S'_\infty(0)} \qquad \text{for all} \quad v \in Z.$$

This implies our conclusions in both cases: For the curves (5.17) we obtain the constancy of the ratio of slopes of their tangents, and for the curves (5.18) we have the constancy of the angle between their tangents.

6. Proof of Proposition (VI): The Case of the Euler Splines

Now $t = -1$ and we use for $S_n(x) = S_n(x; -1)$ the term *Euler splines*, omitting the "exponential." These are well-known functions which are the
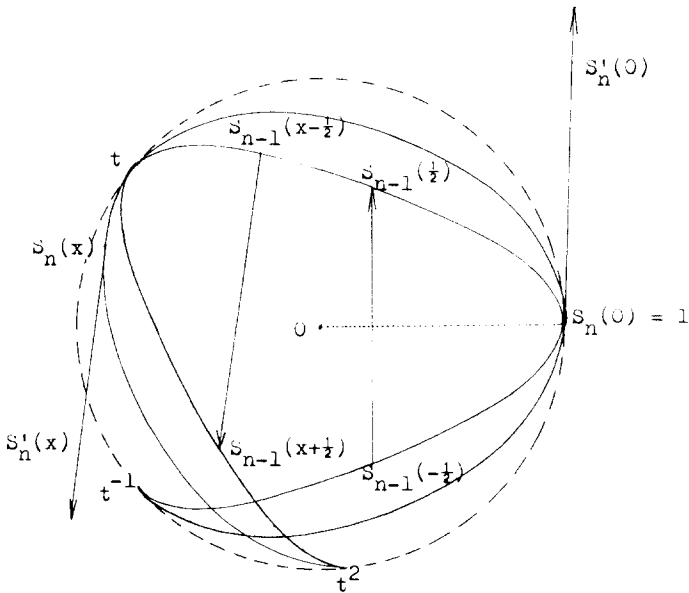

FIGURE 2

subject of Kolmogorov's famous extremum property of the Euler splines (for references see [4]).

Our construction (2.5) is particularly effective in this case and we will actually prove more than (VI) and establish the known proposition.

(VIII) *The function* $S_n(x) = S_n(x; -1)$ *has all the properties of the function* $\cos \pi x$ *concerning symmetries, zeros, signs, and monotonicities.*

Proof. These statements are evident for $n = 1$, when $S_1(x) = E_1(x)$ is the linear Euler spline which the author found so useful in a study of billiard ball motions in a cube [5]. If we assume (VIII) to be true up to and including $n - 1$, then the validity of all parts of (VIII) for the function

$$S_n(x) = \int_{x-1/2}^{x+1/2} S_{n-1}(u)\, du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u)\, du \qquad (6.1)$$

become so obvious, that we may practically omit any further discussion. A few remarks will certainly suffice. The identity $S_n(x + 1) = -S_n(x)$ is proved as in (4.1), and it implies the periodicity $S_n(x + 2) = S_n(x)$. All symmetries and sign properties of $S_{n-1}(x)$ are transmitted to $S_n(x)$ by (6.1), e.g., *why is* $S_n(x)$ *decreasing in* $[0, 1]$? Answer: Because as x increases from 0 to 1, the numerator of (6.1) visibly decreases, since it drops positive area while adding negative area (a rough diagram helps).

Let us sketch a proof of (3.13). Since $S_1(x) = |1 - 2x|$ if $-1 \leqslant x \leqslant 1$, we obtain the Fourier series

$$S_1(x) = \sum_{-\infty}^{\infty} \frac{1}{(2\nu + 1)^2}\, e^{(2\nu + 1)\pi i x} \Big/ \sum_{-\infty}^{\infty} \frac{1}{(2\nu + 1)^2}. \qquad (6.2)$$

Notice also that (5.5), for $\gamma = \pi i$, gives

$$\int_{x-1/2}^{x+1/2} e^{(2\nu + 1)\pi i x}\, dx = \frac{2(-1)^{\nu}}{(2\nu + 1)\, \pi}\, e^{(2\nu + 1)\pi i x}.$$

Performing the operation $\int_{x-1/2}^{x+1/2} (\cdot)\, dx$ on both sides of (7.2) $n - 1$ times and dividing by the value at $x = 0$ of the result, we obtain that

$$S_n(x) = \sum_{-\infty}^{\infty} \frac{(-1)^{\nu(n-1)}}{(2\nu + 1)^{n+1}}\, e^{(2\nu + 1)\pi i x} \Big/ \sum_{-\infty}^{\infty} \frac{(-1)^{\nu(n-1)}}{(2\nu + 1)^{n+1}}.$$

Notice that in both series the coefficients of the terms for $\nu = 0$ and $\nu = -1$ become $= 1$. It follows that we may rewrite the last expansion as

$$S_n(x) = \left\{ e^{\pi i x} + e^{-\pi i x} + \sum_{\nu=0,1} \frac{(-1)^{\nu(n-1)}}{(2\nu+1)^{n-1}} e^{(2\nu+1)\pi i x} \right\} \Big/$$

$$\left\{ 1 + 1 + \sum_{\nu=0,1} \frac{(-1)^{\nu(n-1)}}{(2\nu+1)^{n+1}} \right\}$$

$$= \left\{ 2\cos\pi x + 2\sum_{\nu=1} \frac{1}{(2\nu+1)^{n+1}} \cos(2\nu+1)\pi x \right\} \Big/$$

$$\left\{ 2 + 2\sum_{\nu=1} \frac{1}{(2\nu+1)^{n+1}} \right\},$$

which implies that

$$S_n(x) = \cos\pi x + O\left(\frac{1}{3^{n+1}}\right) \qquad \text{as} \quad n \to \infty.$$

7. A Geometric Corollary of (2.5) for Bounded Exponential Euler Splines

The only bounded exponential t^x have $|t| = 1$, hence $t^x = e^{i\alpha x}$. However, the corresponding sequence of splines $S_n(x; t)$ are the most attractive among exponential Euler splines (see $|3$, Dedicatory page and pp. 29–32$|$). The curve $z = S_n(x; t)$ is *bounded* iff $|t| = 1$, or

$$t = e^{i\alpha}, \qquad \text{with } 0 < \alpha < \pi, \text{ say.} \tag{7.1}$$

From the unicity property of proposition (VII) (Section 6) it easily follows that the curve

$$\Gamma_{n,\alpha} : z = S_n(x) = S_n(x; e^{i\alpha}) \qquad (-\infty < x < \infty) \tag{7.2}$$

has all the rotational and ordinary symmetry properties of the polygon $\Gamma_{1,\alpha} : z = S_1(x; e^{i\alpha})$. Moreover, the curve (7.2) is *closed* iff $t = e^{i\alpha}$ is a root of unity. $\Gamma_{n,\alpha}$ is contained in $|z| < 1$, except at the points $t^\nu = e^{i\nu\alpha}$ which it interpolates. The arcs between two such consecutive points bulge out progressively as n increases, and converge to $|z| = 1$ as $n \to \infty$, by proposition (v) (Section 3). The knots of $\Gamma_{n,\alpha}$ are on $|z| = 1$ if n is odd, or in the midpoints of the arcs if n is even.

Figure 2 represents three arcs of each of the consecutive curves

$$\Gamma_{n-1,\alpha} \qquad \text{and} \qquad \Gamma_{n,\alpha} \quad (n \geqslant 2). \tag{7.3}$$

On these arcs we have marked in Fig. 2 the three points

$$S_n(x), \ S_{n-1}(x + \tfrac{1}{2}), \ S_{n-1}(x - \tfrac{1}{2}).$$

Differentiating our fundamental relation (2.5) we obtain

$$S_n'(x) = c_n(S_{n-1}(x + \tfrac{1}{2}) - S_{n-1}(x - \tfrac{1}{2})),$$

$$\text{where } c_n = 1 \Big/ \int_{-1/2}^{1/2} S_{n-1}(u) \, du. \tag{7.4}$$

We claim that

$$c_n > 0. \tag{7.5}$$

Proof. If we let $x = 0$ in (7.4), we obtain

$$S_n'(0) = c_n(S_{n-1}(\tfrac{1}{2}) - S_{n-1}(-\tfrac{1}{2})) \tag{7.6}$$

and Fig. 2 shows that both $S_n'(0)$ and $S_{n-1}(\tfrac{1}{2}) - S_{n-1}(-\tfrac{1}{2})$ become purely imaginary with positive imaginary parts. The reason: $S_n'(0)$ is obviously vertical and pointing upwards, while $S_{n-1}(\tfrac{1}{2})$ and $S_{n-1}(-\tfrac{1}{2})$ are the midpoints of the two symmetric arcs meeting at $z = 1$. Now (7.5) becomes clear from (7.6). We have just proved our last proposition

(IX) *If we think of x as time, then the velocity vector $S_n'(x)$ is parallel to the vector $S_{n-1}(x + \tfrac{1}{2}) - S_{n-1}(x - \tfrac{1}{2})$ and their ratio is a positive constant.*

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